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An Application of Interpolation Systems to the Finite Element Method

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Abstract. In this work we study a problem in bivariated interpolation associated with the Finite Element Method, using an interpolation technique called "Interpolation Systems".

1. Introduction

A great number of papers, (Cf. [1],[2],[3],[5],[6]), dealing with multivariated interpolation give special attention to the construction of different kinds of "Finite Elements"(in R^2) either of triangular or rectangular shape. In this line, some interesting results are obtained in [3] by A. Le Mehaute.

The Finite Element Method solves, in a local sense, the problem of interpolating a function on a triangulated bounded domain of R^2 with polygonal boundary; that is, on each element of the triangulation, T, a function $P \in P(T)$ is constructed, which is the solution of the interpolation problem:

$$L_i(P) = L_i(f) \quad \text{for every } L_i \in \Sigma(T).$$

It is from this point of view that the study of the $P(T)$-unisolvency of $\Sigma(T)$ and the obtaining of the functions solving the problem become important. Here $P(T)$ is a space of functions included in $C^k(T)$ and $\Sigma(T)$ is a class of linear functionals supported in T. $(T, P(T), \Sigma(T))$ is called a Finite Element of class k.

In this paper we apply the technique of "Interpolation Systems"to this problem. This technique was introduced by M. Gasca and J. I. Maetzu in [2].

2. Notations, Definitions, and Previous Results (see [4])

Let I be a nonempty set of indexes as follows:

$$I = \{(i,j) \in Z_+^2;\ i = 0,\dots,m,\ \ j = 0,\dots,n(i)\} = \{(0,0),\dots,(0,n(0)),\dots,(m,n(m))\}$$

Definition 1. We define an "Interpolation System"(in R^2) as a set S,

$$S = \{(f_i, f_{ij}),\quad (i,j) \in I\}$$

where f_i, f_{ij} are functions from R^2 into R of the type

$$f_i(x,y) = a_i x + b_i y + c_i \quad \text{and} \quad f_{ij}(x,y) = a_{ij}x + b_{ij}y + c_{ij}$$

moreover, the following condition should be verified for each $(i,j) \in I$

$$a_i \cdot b_{ij} \neq a_{ij} \cdot b_i$$

Therefore, there exists a unique point $u_{ij} \in R^2$ such that

$$f_i(u_{ij}) = f_{ij}(u_{ij}) = 0;$$

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Definition 2. Given an Interpolation System, S, we shall say that the set of functions from R^2 into R given below is the "Associated Basis", $B(S)$, for S.

$$B(S) = \{\phi_{ij};\ (i,j) \in I\} \quad \text{where} \quad \phi_{ij} = f_0 \cdot f_1 \cdots f_{i-1} \cdot f_{i0} \cdot f_{i1} \cdots f_{ij-1}$$

(when $i = 0$ or $j = 0$ in the previous product, the corresponding factors take the value 1).

Definition 3. We define the "Associated Data"as the family $L(S)$,

$$L(S) = \{L_{ij};\quad (i,j) \in I\}$$

where each L_{ij} is the functional defined by

$$L_{ij}(f) = \frac{\delta^{s+t} f}{\delta \rho_i^s \delta \rho_{ij}^t}|_{u_{ij}} \qquad \text{with } f : R^2 \to R$$

where $\rho_i = (-b_i, a_i)$, $\rho_{ij} = (-b_{ij} a_{ij})$; and we denote by t (resp. s) the number of functions in the set

$$\{f_0, f_1, \ldots, f_{i-1}, f_{i0}, f_{i1}, \ldots, f_{ij-1}\}$$

vanishing at the point u_{ij} and such that their graphs coincide (resp. do not coincide) with the graph of f_i.

We denote Π_m the real vector space of polynomials in two variables with total degree less or equal than m. It is known that the dimension of Π_m is $\frac{1}{2}(m+1)(m+2)$. The proofs of the following results can be found in [2].

THEOREM 1. *$Det\{L_{ij}(\phi_{hk});\ (i,j);(h,k) \in I\} \neq 0$ Moreover, if the couples (i,j) and (h,k) are arranged according to the lexico-graphical order, the corresponding matrix is triangular.*

Corollary 1. The interpolation problem:
"To find $p \in \langle B(S) \rangle = Sp\{B(S)\}$ satisfying $L_{ij}(p) = W_{ij}$ for each $(i,j) \in I$ with $W_{ij} \in R$"; admits a unique solution.

THEOREM 2. *$\langle B(S) \rangle = \Pi_m(x,y)$ iff $I = \{(i,j) \in Z_+^2;\ 0 \leq i+j \leq m\}$. Moreover, S is called "System of order m"(cf. [2]).*

3. TRIANGULAR FINITE ELEMENT OF CLASS k AND TYPE (k, N, χ, p)

Now we are going to construct an Interpolation System for a particular class of Finite Elements studied by A. Le Mehaute in [3].

Let T be a triangle with vertices A_1, A_2, A_3 and Σ be a set of functionals associated with T in such that $(T, P(T), \Sigma)$ is a Finite Element. Then one can distinguish two types of functionals (or degrees of freedom) in Σ:
a) The functionals given in order to obtain the class of Finite Element are called of First Type. They appear on the vertices and sides of the triangle T (Cf.[6]).
b) The functionals which, jointly with those of First Type, are given to guarantee the $P(T)$ - unisolvency for Σ are called of Second Type.

Definition 4. (see [3]). - We say that $(T, P(T), \Sigma)$ is a Finite Element of Type (k, N, χ, p) if the functionals of the first type are given by

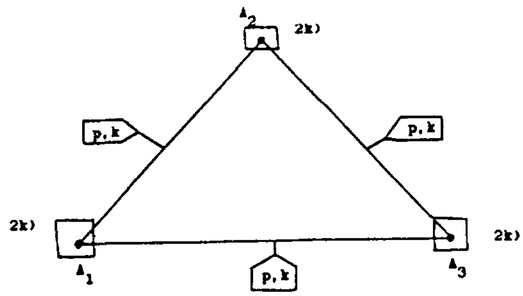
$$\text{a) } l_i(f) \in \{D^\alpha f(A_s) \text{ with } |\alpha| \leq k + \chi,\ s = 1,2,3\} \tag{1}$$

$$\text{b) } l_i(f) \in \left\{ \frac{\delta^j}{\delta \mu_s^{-j}} (Q_{rs}^j,\ s = 1,2,3;\ j = 0, \ldots, k;\ r = 1, \ldots, p+j \right\} \tag{2}$$

where Q^j_{rs} are given points on the side opposite to A_s and such that they are not vertices and depend on r and $\bar{\mu}_m$ is a normal direction to A_s.

Here we shall only describe the case $\chi = k$ and $N = 4k + p + 1$ although the general case can be treated in an analogous way. We shall obtain an Interpolation System with an "associated data"set equivalent to the set given by (1)-(2) and with $\langle B(S)\rangle \subset \Pi_{4k+p+1}$.

Let T be the triangle of Figure 1 with vertices A_1, A_2, A_3 and let R_1, R_2, R_3 be the lines supporting the sides opposite to A_1, A_2, A_3 respectively



(Figure 1)

We define the following set of functions, S, from R^2 into R

$$S = \{(f_i, f_{ij}, u_{ij});\quad (i,j) \in I\}$$

where, for $i = 0, 1, \dots, k$ $\quad f_*, f_{*j}$ are defined as follows:

$$f_{3i} \equiv R_1;\quad f_{3i+1} \equiv R_2;\quad f_{3i+2} \equiv R_3$$

$$f_{3i,j} \equiv \begin{cases} R_2 & j = 0, \dots, 2(k-i); \qquad \text{point} \quad A_3 \\ \mu_{r1} & j = 2(k-i) + r \ con \ r = 1, \dots, p+i; \quad \text{point} \quad Q^i_{r1} \\ R_3 & j = 2k+p+1-i, \dots, 4k+p+1-3i; \quad \text{point} \quad A_2 \end{cases}$$

$$f_{3i+1,j} \equiv \begin{cases} R_3 & j = 0, \dots, 2(k-i); \qquad A_1 \\ \mu_{r2} & j = 2(k-i) + r, \quad r = 1, \dots, p+i \quad Q^i_{r2} \\ R_1 & j = 2k+p+1-i, \dots, 4k+p-3i; \quad A_3 \end{cases}$$

$$f_{3i+2,j} \equiv \begin{cases} R_1 & j = 0, \dots, 2(k-i)-1; \qquad A_2 \\ \mu_{r3} & j = 2(k-i) - 1 + r, \quad r = 1, \dots, p+i \quad Q^i_{r3} \\ R_2 & j = 2k+p-i, \dots, 4k+p-3i-1; \quad A_1 \end{cases}$$

where μ_{rs} denote the line passing through Q_{rs} with direction $\overline{\mu}_s$.

Thus we obtain the following results:

THEOREM 3. *Let P be the space spanned by $B(S)$. Then the following statements hold:*
a) $\Pi_{3k+2} \subset P \subset \Pi_{4k+p+1}$ if $k \geq 1$.
b) $L(S)$ is equivalent to $\underline{\Sigma}'$ as given by (1) and (2).
c) The space P^* spanned by the functions
$\{(R_1 \cdot R_2 \cdot R_3)^{k+1} \cdot \pi_i \quad i = 1, \dots, \frac{1}{2}(k+p-1)(k+p)\}$
is orthogonal to P with respect to $L(S)$. Moreover, $P + P^* \equiv \Pi_{4k+p+1}$. (where Π_i denote the elements of a basis of Π_{k+p-2}).

THEOREM 4. *Let l_i $\ i = 1, \dots, \frac{1}{2}(k+p-1)(k+p)$ be a family of linear functionals -of second type- associated to T and consider the set $\Sigma = L(S) \cup \{l_i\}$. Then, Σ is Π_{4k+p+1} unisolvent if and only if $\{l_i\}$ is P^*-unisolvent.*

REMARK.- *From theorem 3 and theorem 4, $(T, \Pi_{4k+p+1}, \Sigma)$ is a Finite Element of class k of type $(k, 4k+p+1, k, p)$.*

CONCLUSIONS

1.- The construction of the solution of the interpolation problem associated with a Finite Element of type $(k, 4k+p+1, k, p)$ is made by the following process
$P = P_1 + P_2$ where P_1 is constructed in a recurrent way as the solution of the problem

$$L_{ij}(P_1) = L_{ij}(f) \quad \text{for all} \quad L_{ij} \in L(S) \tag{3}$$

and P_2 is the solution of the interpolation problem

$$l_i(P_2) = l_i(f) - l_i(P_1) \quad i = 1, \dots, \frac{1}{2}(k+p)(k+p-1) \tag{4}$$

2.- Once P_1 has been constructed, the system to solve is of order $\frac{1}{2}(k+p)(k+p-1)$; while in [3], once the Taylor polynomial associated to T was constructed, one had to solve a system of order $\frac{1}{2}(k+p)(k+p-1) + \frac{3}{2}(2p+k)(k+1)$.

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